Preconditioning on high-order element methods using Chebyshev–Gauss–Lobatto nodes

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Available online 27 March 2008

Abstract

Based on Chebyshev–Gauss–Lobatto points, the piecewise linear finite element preconditioner is analyzed in terms of condition numbers for the high-order element discretizations applied to a model elliptic operator. The optimality of such a preconditioner is proved for one-dimensional case and the scalability is shown for two-dimensional case. Further, we provide \( O(N^{1/3}) \) growth of piecewise linear finite element preconditioner numerically.

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MSC: 65F10; 65M30

Keywords: Multigrid methods; High-order element methods; Piecewise linear preconditioning; Chebyshev–Gauss–Lobatto nodes

1. Introduction

It is well known that a linear system arisen from the high-order element discretizations has large condition numbers dependent on the mesh sizes and the degrees of high-order elements even for a simple elliptic problem. Hence, high-order element approaches require efficient iterative methods such as Schwarz-based methods (see [6,7,22,24]), preconditioning methods related to multilevel in multigrid algorithms (see [8,12]) and so on. Starting with the finite difference preconditioner suggestion of Orszag [18] for a single spectral element, several authors have studied the use of finite element/difference preconditioner for pseudospectral or spectral methods in [2,4,5,14,15,21,19,20,23], etc. Further, such a preconditioning using Legendre–Gauss–Lobatto (=LGL) nodes was analyzed numerically in [7,16] and theoretically in [13] recently. This kind of piecewise linear preconditioning is very important for the cases of Stokes and elasticity equations because the complexity occurred from the high-order discretizations for multigrid methods is even worse than that of a simple elliptic partial differential equation. The piecewise linear finite element methods corresponding to an elliptic boundary value problem are used as efficient preconditioners for the discretizations by high-order element methods in order to employ preconditioning iterative methods or multigrid preconditioning. In particular, for the usages of geometric multigrid preconditioner using piecewise linear finite elements, the nested property for the class of the even degrees of polynomials is pointed out in [12] as one of advantages.
of the Chebyshev–Gauss–Lobatto (=: CGL) nodes, which makes adaptive p-refinement more straightforward. Hence it is worth to investigate an efficient finite element preconditioner for high-order element discretizations using CGL nodes.

In this paper, both the high-order element discretizations and the finite element preconditioner will be constructed using CGL nodes without the Chebyshev weight function \( w(x) = \frac{1}{\sqrt{1-x^2}} \). Hence we pay attention to analyze the finite element preconditioner using CGL nodes for high-order elements using same nodes. A few articles can be found for the finite element multigrid preconditioner using CGL nodes in [9,14,15] and [25] for spectral collocation methods. It is shown that finite element/difference preconditioners are optimal using Chebyshev quadrature rule (see [9,14,15]).

2. Description of the problem and goal

The following uniformly elliptic operator

\[
Lu := -\nabla \cdot \nabla u + \beta u = f \quad \text{in } \Omega \tag{2.1}
\]

with a nonzero constant \( \beta \) can be chosen for the present discussions of the piecewise linear finite element preconditioner. The boundary conditions are given by

\[
u = 0 \quad \text{on } \Gamma_D, \quad n \cdot \nabla u = 0 \quad \text{on } \Gamma_N\tag{2.2}
\]

where \( \partial \Omega = \Gamma_D \cup \Gamma_N \) with a nonempty \( \Gamma_D \) and \( \Gamma_D \cap \Gamma_N = \emptyset \). Further, we assume for simplicity

\[
\Gamma_D := \{-1\} \times [-1, 1] \cup [-1, 1] \times \{-1\} \quad \text{and} \quad \Gamma_N := \partial \Omega \setminus \Gamma_D. \tag{2.3}
\]

We will consider the high-order element methods for (2.1) employing the piecewise continuous Chebyshev Lagrange polynomials using CGL nodes, which yield the matrix \( \tilde{G} \) for one-dimensional case and \( \tilde{S} \) for two-dimensional case through weak formulations over \( \Omega \) without the Chebyshev weight function. In such a way, the piecewise linear finite element preconditioners \( \tilde{F} \) for one-dimension and \( \tilde{C} \) for two-dimension will be constructed, respectively.

The goal is to provide a validity of \( \tilde{F} \) and \( \tilde{C} \) as a good preconditioner for \( \tilde{G} \) and \( \tilde{S} \) respectively. More precisely, it is to prove that for 1D case the condition numbers of \( \tilde{F}^{-1} \tilde{G} \) are independent of the degrees of high-order elements and the mesh sizes, that is, those are uniformly bounded so that \( \tilde{F} \) is an optimal preconditioner. For 2D case, it is to provide the scalable preconditioner \( \tilde{C} \) analytically. That is, the condition numbers of \( \tilde{C}^{-1} \tilde{S} \) are independent of the number of elements but dependent on the degrees \( N \) of polynomials.

For the numerical implementations, we will approximate \( \tilde{G} \) and \( \tilde{S} \) by \( \tilde{G} \) and \( \tilde{S} \) respectively, which are done through LGL quadrature at LGL nodes. For these computations, we will provide the concrete formulations for differentiation and mass matrices in Appendix A. We demonstrate that the \( N \)-dependency of condition numbers of \( \tilde{C}^{-1} \tilde{S} \) mainly due to two one-dimensional mass matrices \( M^I \) and \( M^C \) from finite element and high-order element discretization, respectively. Also, we provide that condition numbers of \( (M^I)^{-1} M^C \) are dependent on the degrees \( N \) of polynomials with the growth \( O(N^{1/3}) \) numerically for \( N \leq 40 \). Hence \( \tilde{C}^{-1} \tilde{S} \) has the same growth \( O(N^{1/3}) \) of condition numbers as \( (M^I)^{-1} M^C \) has. Under same situations, thanks to anonymous referee’s suggestions, the preconditioner \( \tilde{P} \) (see Section 7.2) presented for spectral collocation schemes in [23] is tested for high-order element discretization, which turns out to be optimal numerically.

The goal of this paper can be achieved by exploring a local interpolation operator \( I_N \) from piecewise linear space to Chebyshev Lagrange polynomial space in terms of \( H^1 \), \( L^2 \)-norm and by extending such estimates to a global interpolation operator \( \tilde{I}_N^h \). In this paper, without the Chebyshev weight function \( w(x) = \frac{1}{\sqrt{1-x^2}} \), it is proven that the piecewise linear function \( u \) at CGL nodes and its interpolant \( \tilde{I}_N^h u \) at CGL nodes are \( H^1 \)-equivalent, that is, there exist two positive absolute constants \( c \) and \( C \) such that for a piecewise linear function \( u \),

\[
c \| u \|_1 \leq \| \tilde{I}_N^h u \|_1 \leq C \| u \|_1.
\]

On the other hand, Kim and Parter in [14] have shown the equivalence of piecewise linear function \( u \) at CGL nodes and its interpolant \( \tilde{I}_N^h u \) at the same nodes with the Chebyshev weight function \( w(x) = \frac{1}{\sqrt{1-x^2}} \) such that

\[
c \| u \|_{1,w} \leq \| \tilde{I}_N^h u \|_{1,w} \leq C \| u \|_{1,w}.
\]
where $\| \cdot \|_{1,u}$ is Chebyshev weighted $H^1$-norm.

This paper consists of as follows. In next section, we recall some known results, piecewise polynomial basis, interpolation operators, etc. In Section 4, we have the results between the norms of the piecewise linear functions using CGL nodes and the norms of its local and global interpolation operators, respectively, which lead to one-dimensional preconditioning results with some numerical evidences in Section 5. Two one-dimensional mass matrices $M^f$ and $M^b$ are compared in numerical point of view to explain the scalability of the bilinear preconditioner for two-dimensional case in Section 6. It is shown in Section 7 that the piecewise bilinear preconditioner $\hat{C}$ does have the same behaviors as the mass matrix $M^f$ has. Also, we tested the suggested preconditioner $\hat{P}$ in [23] which leads to optimal computational results. In Appendix A, the representations of differentiation and mass matrices are provided in terms of the first and second kind of Chebyshev polynomials. Finally, we mention some conclusions in last section.

3. Preliminary

Let $(n^c_k)^N_{k=0}$ and $(\omega^c_k)^N_{k=0}$ be CGL points and CGL weights in $I = [-1, 1]$, respectively, which are known as

$$n^c_k := -\cos \frac{\pi k}{N}, \quad \omega^c_k := \frac{\pi}{N} \quad \text{for } k = 1, 2, \ldots, N - 1.$$

Let $\{t^c_j\}_{j=0}^M$ be the knots in the interval $I$ such that $-1 := t_0 < t_1 < \cdots < t_{M-1} < t_M := 1$. Here $M$ and $N$ denote the number of subintervals of $I = [-1, 1]$ and the degree of Chebyshev polynomial on $I$, respectively. On each subinterval $I_j := [t_{j-1}, t_j]$, the degree of the polynomial will be denoted by $N_j$. For convenience, denote $\sigma(N_k) := \sum_{j=1}^k N_j$ and in particular, for $k = M$ we will denote $\sigma := \sigma(N_M)$, and $\sigma(0) = 0$ for $k = 0$. Let $(\eta^c_k)^N_{k=0}$ and $(\omega^c_k)^N_{k=0}$ be LGL points and weights in $I$ respectively. The LGL points $(\eta^c_k)_{k=0}^N$ are the zeros of $(1 - t^2)L_N(t)$ where $L_N$ is the $N$th Legendre polynomial and the corresponding quadrature weights are given by

$$\omega^c_0 = \omega^c_N := \frac{2}{N(N+1)}, \quad \omega^c_k := \frac{2}{N(N+1)} \frac{1}{[L_N(\eta^c_k)]^2} \quad \text{for } k = 1, 2, \ldots, N - 1.$$

By translations from $I$ to a $j$th subinterval $I_j = [t_{j-1}, t_j]$, we denote $(\xi^q_{k+\sigma(N_{j-1})})_{j=1,k=0}^{M,N_j}$ as the $k$th CGL points for $q = c$ or LGL points for $q = \ell$ in each subinterval $I_j$ for $j = 1, 2, \ldots, M$, where

$$\xi^q_{j,k} := \frac{h_j}{2} \eta^q_k + \frac{1}{2}(t_{j-1} + t_j), \quad h_j = t_j - t_{j-1} \quad (3.1)$$

and the corresponding LGL weights $\{\rho^\ell_{j,k}\}_{j=1,k=0}^{M,N_j}$ are given by

$$\rho^\ell_{j,k} := \frac{h_j}{2} \omega^\ell_k \quad \text{for } j = 1, 2, \ldots, M. \quad (3.2)$$

All CGL or LGL points $(\xi^q_{k})^\sigma_{k=0}$ can be ordered as $-1 := \xi^q_0 < \xi^q_1 < \cdots < \xi^q_\sigma := 1$. Accordingly, the LGL weights $(\rho^\ell_{j,k})^\sigma_{j=1,k=0}$ can be ordered also. With $\xi^q_{M+1,0} := t_M$, note that in (3.1) and (3.2)

$$\xi^q_{j,N_j} = \xi^q_{j+1,0}, \quad \rho^\ell_{j,N_j} = \rho^\ell_{j+1,0} \quad \text{for } j = 1, 2, \ldots, M.$$

Let $\mathcal{P}_k$ be the space of all polynomials $p_k(t)$ defined on $I = [-1, 1]$ whose degrees are less than or equal to $k$ and let $\mathcal{P}_k^h$ be the subspace of $C(I)$, which consists of piecewise polynomials $p_{N_j}(t)$ with support $I_j = [t_{j-1}, t_j]$ whose degrees are less than or equal to $N_j$ for $j = 1, 2, \ldots, M$. For the reference space $\mathcal{P}_N$ on $I$, the Lagrange basis functions are denoted as $\{\phi_i(t)\}_{i=0}^N$ satisfying

$$\delta_i(\eta^c_k) = \delta_{i,k} \quad \text{for } i, k = 0, 1, \ldots, N,$$

where $\delta$ denotes the Kronecker delta function. By translation, the basis for the space $\mathcal{P}_N^h$ is given by piecewise Lagrange polynomial basis $\{\phi_{k+\sigma(N_{j-1})})_{j=1,k=1}^{M,N_j-1} \cup \{\phi_0\} \cup \{\phi_{N_j}\}$ and $\{\phi_\sigma(N_j) := \phi_{j,N_j}(t)\}_{j=1}^{M-1}$, which satisfy
\[ \phi_0(\xi^r_j) = \delta_{0,\mu}, \quad \mu = 0, 1, \ldots, N_1, \]
\[ \phi_{\mu l}(\xi^r_j) = \delta_{\mu l,\mu}, \quad \mu = \sigma(N_{M-1}), \quad \sigma(N_{M-1}) + 1, \ldots, \Omega_l, \]
\[ \phi_{\nu l}(\xi^r_j) = \delta_{\nu l,\mu}, \quad \mu = k + \sigma(N_{i-1}), \quad v = l + \sigma(N_{i-1}), \]
where \( k = 0, 1, \ldots, N_i, l = 1, 2, \ldots, N_i - 1, i = 1, 2, \ldots, M \) and
\[ \phi_{\sigma(N_j)}(\xi^r_j) = \delta_{\sigma(N_j),\mu}, \quad j = 1, 2, \ldots, M - 1, \quad \mu = \sigma(N_{j-1}), \quad \sigma(N_{j-1}) + 1, \ldots, \sigma(N_{j+1}). \]

Let \( V_N \) be the reference space consisting of all piecewise linear functions \( \hat{\psi}_k(t) \) defined on \( I = [-1, 1] \) satisfying \( \hat{\psi}_k^1(t_i) = \delta_{i,k} \) and define \( P^N_0 \) as the space of all Lagrange piecewise linear functions \( \psi_k(t) \) satisfying \( \psi_k(\xi^r_j) = \delta_{\mu,v} \) for \( \mu, v = 0, 1, \ldots, \Omega \).

Define two interpolation operators \( I_N : C(I) \to P_N(I) \) such that \( (I_N u)(\eta^r_k) = u(\eta^r_k) \) for \( u \in C(I) \) and \( I^h_N : C(I) \to P^h_N(I) \) such that
\[ (I^h_N v)(\xi^r_j) = v(\xi^r_j) \quad \text{for} \ v \in C(I). \]

Because all arguments below can be extended to any odd degrees of polynomials via a minor modification, we assume even degree \( N \) of polynomials \( p_N \). First we recall Clenshaw–Curtis (=:CC) quadrature rule, which is exact up to degree \( 2N \) using \( 2N + 1 \) CGL points \( \{ \xi_j := -\cos \frac{\pi j}{2N} \}^{2N}_{j=0} \)
\[
\int_{-1}^{1} p_{2N}(t) \ dt = \sum_{j=0}^{2N} \tilde{w}_j p_{2N}(\xi_j) \tag{3.3}
\]
where for \( j = 1, 2, \ldots, 2N - 1 \),
\[ \tilde{w}_0 = \tilde{w}_{2N} := \frac{1}{2N(2N+1)}, \quad \tilde{w}_j := \frac{1}{N} \left( 1 - \sum_{k=1}^{N} \frac{2 \cos \frac{\pi jk}{N}}{4k^2 - 1} \right). \]

For two-dimensional case, we will use the direction notation \( t, t = x \) or \( y \). Let \( M^t \) be the number of subintervals of \( t \)-direction and let \( \Omega^t := \sigma(N_{M^t}) \). It will be denoted \( N^t_j \) to represent the degree of polynomial applied on each subinterval \( I^t_j := [t_{j-1}, t_j], j = 1, 2, \ldots, M^t \). Note that \( P^h_{N^t} \) and \( V^h_{N^t} \) are the spaces corresponding to \( P^h_N \) and \( V^h_N \) of \( t \)-direction, respectively. Let \( [P^h_N]^2 := P^h_{N^x} \otimes P^h_{N^y} \) and \( [V^h_N]^2 := V^h_{N^x} \otimes V^h_{N^y} \), whose basis functions are given by tensor products \( \{ \phi_{\mu}(x,y) = \phi_k(x)\phi_l(y) \} \), \( 0 \leq k \leq \Omega^x, 0 \leq l \leq \Omega^y \} \) and \( \{ \psi_{\mu}(x,y) = \psi_k(x)\psi_l(y) \} \), \( 0 \leq k \leq \Omega^x, 0 \leq l \leq \Omega^y \}, \) respectively. Define a discrete inner product \( (u, v)_{V^h_N} := \sum_{\nu=0}^{\Omega^x} \sum_{\mu=0}^{\Omega^y} u(\xi^x_\mu, \xi^y_\nu) v(\xi^x_\mu, \xi^y_\nu) \rho^x_\mu \rho^y_\nu \) and it can be modified properly for one-dimensional case.

Finally, the notation \( a \sim b \) for any two real quantities \( a \) and \( b \) is meant by that there are two positive constants \( c \) and \( C \), which do not depend on the mesh sizes and the degrees of polynomials such that \( 0 < c \leq \frac{a}{b} \leq C < \infty \).

The notation \( (U, V) \) stands for \( \sum u_i v_i \) for any two vectors \( U = (u_1, \ldots, u_d)^T \) and \( V = (v_1, \ldots, v_d)^T \) where the superscript \( T \) denotes the transpose of a vector or matrix. The standard Sobolev spaces \( H^m \) and \( L^2 \) will be used. And we will use \( \| \cdot \|_m \), \( | \cdot |_1 \) and \( \| \cdot \|_0 \) as the standard Sobolev \( H^m \)-norm, \( H^1 \)-seminorm and the usual \( L^2 \)-norm, respectively.

**4. Basic estimates**

Now let us turn to the estimates of interpolation operator \( I_N \) in the sense of \( L^2 \)- and \( H^1 \)-norms.

**Lemma 4.1.** For all \( \phi \in H^m(I), m \geq 1 \), there exists a positive constant \( C \) independent of \( N \) such that
\[ \| \phi - I_N \phi \|_1 \leq CN^{-m} \| \phi \|_m \quad \text{for} \ l = 0, 1. \]

**Proof.** See Lemma 3.3 in [17]. \( \square \)

The following result is an easy consequence of Lemma 4.1.
Corollary 4.2. For all \( \phi \in H^1(I) \), we have
\[
\| \mathcal{I}_N \phi \|_1 \leq C \| \phi \|_1,
\]
where \( C \) is a positive constant independent of \( N \).

In order to discuss one-dimensional piecewise linear finite element preconditioner, it may be required to analyze
the relations between \( \mathcal{I}_N^f \psi \in \mathcal{P}_N^f \) and \( \psi \in \mathcal{V}_N \) in the sense of \( H^1 \)-seminorm, norm and \( L^2 \)-norm. For this purpose, we
will compare \( \mathcal{I}_N \hat{\psi} \in \mathcal{P}_N \) and \( \hat{\psi} \in \mathcal{V}_N \) in \( H^1 \)-seminorm sense first.

Proposition 4.3. It follows that for all \( \hat{\psi} \in \mathcal{V}_N \),
\[
c \| \hat{\psi} \|_1 \leq \| \mathcal{I}_N \hat{\psi} \|_1 \leq C \| \hat{\psi} \|_1,
\]
where \( c \) and \( C \) are absolutely positive constants independent of \( N \).

Proof. For convenience, let \( \psi_N = \mathcal{I}_N \hat{\psi} \) for \( \hat{\psi} \in \mathcal{V}_N \). For the lower bound of (4.1), note that for \( t \in (\eta_j^c, \eta_{j+1}^c) \), it follows that
\[
\hat{\psi}'(t) = \frac{\hat{\psi}(\eta_{j+1}^c) - \hat{\psi}(\eta_j^c)}{z_j} = \frac{\psi_N(\eta_{j+1}^c) - \psi_N(\eta_j^c)}{z_j}, \quad z_j = \eta_{j+1}^c - \eta_j^c.
\]
Hence the result comes from using Fundamental Theorem of Calculus and Cauchy–Schwarz inequality. For the upper
bound of (4.1), let \( C_0 \) be a constant such that \( \hat{\psi} = \hat{\psi} - C_0 \) and \( \int_0^1 \hat{\psi}^2 dt = 0 \). Then we have \( \mathcal{I}_N \hat{\psi} = \mathcal{I}_N \hat{\psi} - C_0 \).
Therefore, using Corollary 4.2 for \( \hat{\psi} \in H^1(I) \) and Poincare’s inequality, it follows that
\[
\| \mathcal{I}_N \hat{\psi} \|_1 = \| \mathcal{I}_N \hat{\psi} \|_1 \leq \| \mathcal{I}_N \hat{\psi} \|_1 \leq c_1 \| \hat{\psi} \|_1 \leq C \| \hat{\psi} \|_1 = C \| \hat{\psi} \|_1.
\]
These arguments complete the proof. \( \square \)

To approximate CC weights, we will use the following formula.

Lemma 4.4. For noninteger \( \alpha \) and \( 2m \pi \leq x \leq (2m + 2) \pi \),
\[
\sum_{k=1}^\infty \frac{\cos kx}{k^2 - \alpha^2} = \frac{1}{2 \alpha^2} - \frac{\pi \cos[\alpha((2m+1) \pi - x)]}{2 \alpha \sin \alpha \pi}.
\]

Proof. See ([11], p. 48). \( \square \)

Lemma 4.5. Let \( \{\tilde{w}_j\}_{j=0}^{2N} \) be the Clenshaw–Curtis weights which make (3.4) exact for \( p_{2N} \) and let \( \{z_j := \eta_{j+1}^c - \eta_j^c\}_{j=0}^{N-1} \) where \( \eta_j^c = -\cos \frac{\pi j}{N} \) are the CGL points. Then it follows that
\[
\tilde{w}_0 \sim z_0, \quad \tilde{w}_{2N} \sim z_{N-1}
\]
and
\[
\tilde{w}_{2j} \sim z_{j+1} + z_j \quad \text{for } j = 1, 2, \ldots, N - 1.
\]

Proof. Since \( \tilde{w}_0 = \tilde{w}_{2N} = \frac{1}{2N(2N+1)} \) and \( z_0 = z_{N-1} = 1 - \cos \frac{\pi}{N} \sim \frac{1}{N^2} \), (4.2) holds. In order to show (4.3), consider for \( j = 1, 2, \ldots, N - 1 \),
\[
\tilde{w}_{2j} = \frac{1}{N} \left( 1 - \sum_{k=1}^N \frac{2}{4k^2 - 1} \cos \frac{2\pi j k}{N} \right)
\]
and
\[
z_{j+1} + z_j = \cos \frac{\pi (j - 1)}{N} - \cos \frac{\pi (j + 1)}{N} = 2 \sin \frac{\pi j}{N} \sin \frac{\pi}{N}.
\]
For the case \( m = 0 \) and \( \alpha = \frac{1}{2} \) in Lemma 4.4 so that \( 0 \leq x = \frac{2\pi j}{N} \leq 2\pi \) where \( j = 1, 2, \ldots, N-1 \), we get

\[
\sum_{k=1}^{\infty} \frac{2}{4k^2-1} \cos \frac{2\pi j k}{N} = \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k^2 - 1/4} \cos \frac{2\pi j k}{N} = 1 - \frac{\pi}{2} \sin \frac{\pi j}{N},
\]

so that

\[
\sum_{k=1}^{N} \frac{2}{4k^2-1} \cos \frac{2\pi j k}{N} = \left( 1 - \frac{\pi}{2} \sin \frac{\pi j}{N} \right) - \sum_{k=N+1}^{\infty} \frac{2}{4k^2-1} \cos \frac{2\pi j k}{N}.
\]

(4.6)

Let us estimate the second term in (4.6). For this, note that the fact \( \frac{2}{N} \leq \sin \frac{\pi j}{N} < \frac{\pi}{N} \) leads to

\[
\sum_{k=N+1}^{\infty} \frac{2}{4k^2-1} \cos \frac{2\pi j k}{N} \leq \frac{1}{4} \sin \frac{\pi j}{N} \quad j = 1, 2, \ldots, N-1
\]

and

\[
- \sum_{k=N+1}^{\infty} \frac{2}{4k^2-1} \cos \frac{2\pi j k}{N} \leq \sum_{k=N+1}^{\infty} \frac{2}{4k^2-1} \cos \frac{2\pi j k}{N} \leq \sum_{k=N+1}^{\infty} \frac{2}{4k^2-1}.
\]

Hence the second term of (4.6) can be estimated as

\[
\frac{1}{4} \sin \frac{\pi j}{N} \leq \sum_{k=N+1}^{\infty} \frac{2}{4k^2-1} \cos \frac{2\pi j k}{N} \leq \frac{1}{4} \sin \frac{\pi j}{N}.
\]

(4.7)

Therefore, combining (4.6) and (4.7), we have

\[
\frac{2\pi - 1}{4} \sin \frac{\pi j}{N} \leq 1 - \sum_{k=1}^{N} \frac{2}{4k^2-1} \cos \frac{2\pi j k}{N} \leq \frac{2\pi + 1}{4} \sin \frac{\pi j}{N}.
\]

Since \( \sin \frac{\pi j}{N} \sim \frac{1}{N} \), comparing (4.4) with (4.5) we have

\[\tilde{w}_{2j} \sim z_{j-1} + z_j \quad \text{for } j = 1, 2, \ldots, N-1.\]

These arguments complete the proof. \( \square \)

Modifying Lemma 7.1 in [26], we have the estimate for the piecewise linear functions.

**Lemma 4.6.** For \( \hat{\psi} \in \mathcal{V}_N \),

\[
\| \hat{\psi} \|_0 \leq \frac{z_0}{2} \hat{\psi}^2(\eta_0^c) + \frac{z_{N-1}}{2} \hat{\psi}^2(\eta_{N}^c) + \sum_{j=1}^{N-1} \frac{z_{j-1} + z_j}{2} \hat{\psi}^2(\eta_j^c) \leq 3 \| \hat{\psi} \|_0,
\]

where \( z_j = n_{j+1}^c - n_j^c \).

The \( L^2 \)-norm of \( \hat{\psi} \in \mathcal{V}_N \) and \( \mathcal{I}_N \hat{\psi} \) can be compared. For one-dimensional case, we need the following proposition. The opposite direction of inequality will be dealt in Section 7.

**Proposition 4.7.** It follows that for all \( \hat{\psi} \in \mathcal{V}_N \),

\[
\| \hat{\psi} \|_0 \leq C \| \mathcal{I}_N \hat{\psi} \|_0
\]

where \( C \) is an absolutely positive constant independent of \( N \).
Proof. Using the Clenshaw–Curtis quadrature rule for a polynomial $|I_N \hat{\psi}(t)|^2$ of the degree $2N$ with quadrature nodes $\xi_k = -\cos \frac{k\pi}{2N}$ so that $\xi_{2k} = \eta_k$, one may have

$$\|I_N \hat{\psi}\|_0^2 = \int_{-1}^{1} |I_N \hat{\psi}(t)|^2 dt = \sum_{k=0}^{N} |I_N \hat{\psi}(\xi_{2k})|^2 \bar{w}_{2k} + \sum_{k=0}^{N-1} |I_N \hat{\psi}(\xi_{2k+1})|^2 \bar{w}_{2k+1} \geq \sum_{k=0}^{N} |I_N \hat{\psi}(\eta_k)|^2 \bar{w}_{2k}.$$ 

Hence, due to the Lemma 4.5 and 4.6, it follows that

$$\|\hat{\psi}\|_0^2 \leq C \left( z_0 \hat{\psi}(\eta_0) + z_{N-1} \hat{\psi}(\eta_N) + \sum_{j=1}^{N-1} (z_{j-1} + z_j) \hat{\psi}(\eta_j) \right) \leq C \|I_N \hat{\psi}\|_0^2.$$ 

It completes the proof. □

We will discuss some estimates of the global interpolation operator $I_N^h$ and in terms of $H^1$- and $L^2$-norms. For $t \in [-1, 1], s \in [t_{j-1}, t_j]$ and a function $u \in H^1$, let

$$v_j(t) := u(s) = u \left( \frac{h_j}{2} t + \frac{1}{2} (t_{j-1} + t_j) \right), \quad h_j = t_j - t_{j-1}, \quad j = 1, 2, \ldots, M. \tag{4.8}$$

**Theorem 4.8.** For all $u \in V_N^h$, we have

$$\|u\|_0 \leq C \|I_N^h u\|_0$$

where $C$ is a constant independent of the mesh sizes and the degrees of polynomials.

**Proof.** Since, for functions $v_j \in V_{N_j}$ and $u \in V_N^h$ with the notation (4.8), we have

$$\|u\|_0^2 = \sum_{j=1}^{M} \frac{h_j}{2} \|v_j\|_0^2 \quad \text{and} \quad \|I_N^h u\|_0^2 = \sum_{j=1}^{M} \frac{h_j}{2} \|I_N v_j\|_0^2. \tag{4.9}$$

Proposition 4.7 yields

$$\sum_{j=1}^{M} \frac{h_j}{2} \|v_j\|_0^2 \leq C \sum_{j=1}^{M} \frac{h_j}{2} \|I_N v_j\|_0^2 = C \|I_N^h u\|_0^2,$$

where $C$ is a constant independent of $h_j$ and $N_j$ for $j = 1, 2, \ldots, M$. Therefore, the proof is completed. □

**Proposition 4.9.** There are positive constants $c$ and $C$ independent of the mesh sizes and the degrees of polynomials such that for all $u \in V_N^h$,

$$c |u|_1 \leq |I_N^h u|_1 \leq C |u|_1.$$

**Proof.** By a change of variables, using the notation (4.8), we have

$$|u|_1^2 = \sum_{j=1}^{M} \int_{t_{j-1}}^{t_j} |u'(s)|^2 ds = \sum_{j=1}^{M} \frac{2}{h_j} \int_{-1}^{1} |v_j'(t)|^2 dt.$$
and
\[ |I_N^h u|^2 = \sum_{j=1}^M \int_{t_{j-1}}^{t_j} |(I_N^h u)'(s)|^2 \, ds = \sum_{j=1}^M \frac{2}{h_j} \int_{t_{j-1}}^{t_j} |(I_N^h v_j)'(t)|^2 \, dt. \]

Then, using Proposition 4.3, we have the conclusion. □

**Theorem 4.10.** There are positive constants \( c \) and \( C \) independent of the mesh sizes and the degrees of polynomials such that for all \( u \in V_N^h \),
\[ c \| u \|_1 \leq \| I_N^h u \|_1 \leq C \| u \|_1. \]

**Proof.** First, note that for \( u \in V_N^h \),
\[ \| I_N^h u \|_1^2 = \| I_N^h u \|_0^2 + |I_N^h u|_1^2 = \sum_{j=1}^M \frac{h_j}{2} \| I_N^j v_j \|_0^2 + \sum_{j=1}^M \frac{2}{h_j} |I_N^j v_j|_1^2. \]

From Lemma 4.1, we recall that for \( v_j \in V_N^j \subset H^1(I) \),
\[ \| I_N^j v_j \|_0^2 \leq \| v_j \|_0^2 + C(N_j)^{-2} \| v_j \|_1^2. \]

Then, using Proposition 4.3 and the fact \( N_j \geq 1, j = 1, 2, \ldots, M \), we have
\[ \| I_N^h u \|_1^2 \leq \sum_{j=1}^M \frac{h_j}{2} (\| v_j \|_0^2 + C(N_j)^{-2} \| v_j \|_1^2) + \sum_{j=1}^M \frac{2}{h_j} |v_j|_1^2 \]
\[ \leq C \left( \sum_{j=1}^M \frac{h_j}{2} \| v_j \|_0^2 + \sum_{j=1}^M \frac{2}{h_j} |v_j|_1^2 \right) \]
\[ = C (\| u \|_0^2 + |u|_1^2) = C \| u \|_1^2, \]
which leads to the upper bound. Theorem 4.8 and Proposition 4.9 yield the lower bound. Hence these arguments complete the proof. □

**5. Optimal preconditioner for 1D case**

Consider the differential operators given by
\[ B^k u := -u'' + b_k u \quad \text{in } I = (-1, 1), \quad k = 1, 2 \] (5.1)
with boundary conditions
\[ u(-1) = u(1) = 0 \quad \text{or} \quad u(-1) = u'(1) = 0, \quad (5.2) \]
where the coefficients \( b_k \) are nonnegative constants. Associated with the operators \( B^k \) are bilinear forms \( B_k(\cdot, \cdot) \), which lead to the variational forms
\[ B_k(u, v) := (u', v') + (b_k u, v) \]
for both the high-order element methods and the finite element methods of the operators \( B^k \). For \( n = 0, 1 \), let \( V^n \) be the subspace of \( H^n(I) \), where \( H^1(I) \) stands for \( H^1(I) \), such that
\[ V^n := \{ u \in H^1(I): u(-1) = u^{(0)}(1) = 0 \}, \]
where \( u^{(0)}(1) \) and \( u^{(1)}(1) \) mean \( u(1) \) and \( u'(1) \), respectively.

Let \( V_{h,N}^n \) and \( V_{h,N}^h \) be the subspaces of \( V^n \) such that
\[ V_{h,N}^n := V^n \cap V_{h,N}^h \quad \text{and} \quad V_{h,N}^h := V^n \cap V_{h}^h \] (5.3)
whose suitable basis functions \( \{ \phi_\mu \}_{\mu=1}^d \) and \( \{ \psi_\nu \}_{\nu=1}^d \) can be given respectively, where
\[
d := \dim(\mathcal{P}_n^{h,N}) = \dim(\mathcal{V}_n^{h,N}).
\]
The variational forms of the finite element methods and high-order element methods are same as \( B_k(\cdot, \cdot) \) on the spaces \( \mathcal{V}_n^{h,N} \) and \( \mathcal{P}_n^{h,N} \), respectively, on which these bilinear forms induce operators \( F \) and \( G \), that is,
\[
F : \mathcal{V}_n^{h,N} \to \mathcal{V}_n^{h,N}, \quad G : \mathcal{P}_n^{h,N} \to \mathcal{P}_n^{h,N}
\]
with
\[
(F f, g) = B_2(f, g) \quad \text{for all } f, g \in \mathcal{V}_n^{h,N}
\]
and
\[
(G u, v) = B_1(u, v) \quad \text{for all } u, v \in \mathcal{P}_n^{h,N}.
\]
Let \( \widehat{F} \) and \( \widehat{G} \) be the matrix representations of \( F \) and \( G \) respectively, whose elements can be written as
\[
\widehat{G}(\mu, \nu) = B_1(\phi_\mu, \phi_\nu), \quad \widehat{F}(\mu, \nu) = B_2(\psi_\mu, \psi_\mu).
\]
Further, we denote \( S^k \) and \( S^f \) as the stiffness matrices and let \( M^s \) and \( M^f \) be the mass matrices such that
\[
S^k(\mu, \nu) = (\phi_\mu, \phi_\nu), \quad S^f(\mu, \nu) = (\psi_\mu, \psi_\nu)
\]
and
\[
M^s(\mu, \nu) = (\phi_\mu, \phi_\nu), \quad M^f(\mu, \nu) = (\psi_\mu, \psi_\nu).
\]
Then we have
\[
\widehat{G} = S^k + b_1 M^s, \quad \widehat{F} = S^f + b_2 M^f.
\]
Now we can discuss one-dimensional preconditioning results. For \( u \in \mathcal{V}_n^{h,N} \), it can be written as
\[
u(t) = \sum_{\mu=1}^d u_\mu \psi_\mu(t)
\]
and its piecewise polynomial interpolation \( \mathcal{I}_N^h u \in \mathcal{P}_n^{h,N} \) can be represented as
\[
(\mathcal{I}_N^h u)(t) = \sum_{\mu=1}^d u_\mu \phi_\mu(t).
\]

**Theorem 5.1.** The eigenvalues \( \{ \lambda_\mu \}_{\mu=1}^d \) of \( \widehat{F}^{-1} \widehat{G} \) are all positive real and bounded above and below. The bounds are independent of the mesh sizes \( h \) and the degrees of the polynomials \( N \). That is, there are positive constants \( c \) and \( C \) independent of \( h \) and \( N \) such that
\[
0 < c \leq \lambda_\mu \leq C < \infty.
\]

**Proof.** Since the matrices \( \widehat{F} \) and \( \widehat{G} \) are symmetric and positive definite, all eigenvalues are real and positive. Hence \( \widehat{F}^{-1} \widehat{G} \) has all positive real eigenvalues. Now let \( U = (u_1, u_2, \ldots, u_d)^T \) be the coefficients vector of \( u \) in (5.5) and \( \mathcal{I}_N^h u \) in (5.6). Then definitions of bilinear forms yield that
\[
(\widehat{G} U, U) = B_1(\mathcal{I}_N^h u, \mathcal{I}_N^h u) \sim \| \mathcal{I}_N^h u \|_1^2,
\]
and
\[
(\widehat{F} U, U) = B_2(u, u) \sim \| u \|_1^2.
\]
Theorem 4.10 gives
\[
(\widehat{F} U, U) \sim (\widehat{G} U, U).
\]
Hence the conclusions come through the min-max theorem.
5.1. Numerical example

Now consider one-dimensional elliptic operator such that

\[ Lu := -u'' + u \quad \text{in } I = (-1, 1) \]

with the boundary condition

\[ u(-1) = u'(1) = 0. \]

For actual computations, one has to compute \( \hat{G} \) using LGL quadrature rule at LGL nodes. For this, let the matrices \( D, W \) and \( M \) be defined as

\[
D_{ij} = \phi_j'(\xi_i^f), \quad W = \text{diag}(\omega_i^f), \quad M_{ij} = \phi_j(\xi_i^f), \quad i, j = 1, 2, \ldots, d. \tag{5.8}
\]

Note that \( \xi_i^f \) denotes the one-dimensional LGL points. Therefore the matrix \( M \) cannot be the identity matrix because the representation nodes are different from one that should be evaluated. The differentiation matrix \( D \) and the mass matrix \( M \) can be represented in an easy form, which are given in Appendix A for reader’s convenience. The matrices \( S^f \) and \( M^f \) defined by polynomials using CGL nodes can be approximated due to LGL quadrature rule so that

\[
(S^f U, U) \sim ((D^T WD) U, U), \quad (M^f U, U) \sim ((M^T WM) U, U). \tag{5.9}
\]

Therefore, the matrix \( \hat{G} \) can be replaced by the equivalent \( \tilde{G} \) for practical computations because

\[
B_1(u, u) \sim \langle u', u' \rangle_{\mathcal{H}} + \langle u, u \rangle_{\mathcal{H}} = (\tilde{G} U, U) \quad \text{where} \quad \tilde{G} := D^T WD + M^T WM.
\]

We compute the eigenvalues of the preconditioned matrix \( \tilde{F}^{-1} \tilde{G} \) where \( \tilde{G} = D^T WD + M^T WM \) with preconditioner matrix \( \tilde{F} = S^f + M^f \) and provide the condition numbers of those. As noticed, all condition numbers are bounded independent of the mesh sizes and the degrees of polynomials. These facts support the theory developed, which are well represented in Table 1. The numeric results are given by the same mesh sizes \( M = 2, 4, 8, 16, 32 \) and the same degrees of the polynomials \( N = 4, 8, 12, \ldots, 40 \) on each element. Fig. 1 explains the growth of the ratios but it reveals that the ratios converge to 1, which means that condition numbers do not grow at all. The graphs of Fig. 1 show that the ratios of the condition numbers for \( (M, N + 4) \) to the ones for \( (M, N) \) for each fixed \( M = 2, 4, 8, 16, 32 \) and the ratios of the condition numbers for \( (M + 4, N) \) to the ones for \( (M, N) \) for each fixed \( N = 2, 4, 8, 16, 32 \), respectively.

6. Two mass matrices

Prior to discussions of piecewise bilinear preconditioner in next section, we first consider the comparison of two mass matrices over the spaces \( \mathcal{V}_{h,N}^n \) and \( \mathcal{P}_{h,N}^n \) because the behavior of the piecewise bilinear preconditioner mainly depends on the properties of these two mass matrices.
First of all, we consider the condition numbers of \((M_f)^{-1} M_s\) numerically for \(N \leq 40\) for a single element. In the left of Fig. 2, it is shown that the condition numbers \(\kappa((M_f)^{-1} M_s)\) grows as \(N\) increases \((N \leq 40)\) and the ratios \(\frac{\kappa((M_f)^{-1} M_s)}{N^{1/3}}\) seem to be bounded. Further, from the numeric results shown in right of Fig. 2, we may guess the behavior of condition numbers such that

\[
2.2N^{1/3} \leq \kappa((M_f)^{-1} M_s) \leq 2.6N^{1/3}.
\]

As a result, one may notice that the piecewise bilinear finite element preconditioner does depend only on the effects of \(M_f\) to \(M_s\) in next section. Unfortunately, this phenomenon is not analyzed thoroughly in this paper. Instead, we can show only the dependency of the degree \(N\) of a polynomial analytically.
7. Preconditioners for 2D case

In this section, we provide that the bilinear finite element preconditioner \( \widehat{C} \) for \( \widehat{S} \) is scalable in the sense that the condition numbers of \( \widehat{C}^{-1} \widehat{S} \) do not depend on the mesh sizes. For this, we will consider the elliptic problem (2.1)–(2.2) defined in \( \Omega \subset \mathbb{R}^2 \) with \( \beta = 2 \) and (2.3).

Define, with the notation (5.3), \([P_{h,N}^n]^2 := P^n_{h_x,N_x} \otimes P^n_{h_y,N_y} \) and \([V_{h,N}^n]^2 := V^n_{h_x,N_x} \otimes V^n_{h_y,N_y} \) and \(d^{xy} := \dim([P_{h,N}^n]^2) = \dim([V_{h,N}^n]^2)\). Note that \(d^{xy} = d^x d^y\), where \(d^i := \dim(P^n_{h_i,N_i}) = \dim(V^n_{h_i,N_i}), \) \(i = x \) or \(y\). Let us order the CGL or LGL points by horizontal lines and we list all the points respectively as \(\{2^q_i\}_{i=1}^{d^{xy}} := \{(\xi_\mu, \xi_\nu)\}_{\mu=1}^{d^x}, \nu=1\) where \(P = \mu + (v - 1)d^x\) for \(\mu = 1, 2, \ldots, d^x, \) \(v = 1, 2, \ldots, d^y\).

Let \(\{\widehat{S} := \widehat{S}_{h^2,N^2}\}\) be a family of high-order element discretizations based on CGL points using the Lagrange basis \(\{\Phi_\mu(x, y) = \phi_\mu(x) \phi_\nu(y)\}\) on the space \([P_{h,N}^n]^2\) arisen from a variational representation of the operator \(L\) in (2.1) and (2.2)

\[
B(u, v) = \int_{\Omega} \nabla u \cdot \nabla v + 2uv \, dx \, dy \quad \text{for } u, v \in [P_{h,N}^n]^2.
\]  

Then the high-order element approach for (2.1) leads to

\[
\widehat{S}U = F, \quad \text{where } \widehat{S}(\mu, \nu) = B(\Phi_\mu, \Phi_\nu).
\]  

The matrix \(\widehat{S}\) can be written as

\[
\widehat{S} = M_x^f \otimes \widehat{G}_x + \widehat{G}_y \otimes M_y^f, \quad \widehat{G}_t = S_t^f + M_t^f,
\]  

where \(S_t^f\) and \(M_t^f\) denote the one-dimensional high-order stiffness and mass matrices respectively.

Let \(\{\widehat{C} := \widehat{C}_{h^2,N^2}\}\) be a family of piecewise bilinear finite element discretizations based on CGL points using the Lagrange basis \(\{\Psi_\mu(x, y) = \psi_\mu(x) \psi_\nu(y)\}\) on \([V_{h,N}^n]^2\) corresponding to (7.1). Then its matrix representation of \(B(\cdot, \cdot)\) becomes

\[
\widehat{C} = M_x^f \otimes \widehat{F}_x + \widehat{F}_y \otimes M_y^f, \quad \widehat{F}_t = S_t^f + M_t^f,
\]  

where \(S_t^f\) and \(M_t^f\) denote the one-dimensional low-order stiffness and mass matrices respectively. For \(u(x, y) = \sum_{\nu=1}^{d^y} \sum_{\mu=1}^{d^x} u_{\mu\nu} \psi_\mu(x) \psi_\nu(y) \in [V_{h,N}^n]^2\), we have

\[
{T}_N^h u(x, y) = \sum_{\nu=1}^{d^y} \sum_{\mu=1}^{d^x} u_{\mu\nu} \phi_\mu(x) \phi_\nu(y)
\]  

where \(T_N^h\) is understood as two-dimensional version of interpolation operator defined in (3.3) at two-dimensional CGL points. Let \(U\) be the coefficients vector of \(u\) in (5.5) and \(T_N^h u\) in (5.6). Due to Theorem 5.1, we have

\[
(\widehat{F}_t U, U) \sim (\widehat{G}_t U, U) \quad \text{where } t = x, y.
\]  

Before proceeding to further, we consider the ratios of CGL weight \(\omega^C_k\) and CC weight \(\bar{\omega}_{2k}\):

\[
\sigma_k = \frac{\omega^C_k}{\bar{\omega}_{2k}} \quad \text{for } k = 0, 1, \ldots, N,
\]  

which are defined in Lemma 4.5. From (4.4) and the boundedness of trigonometric function, we have

\[
\sigma_0 = \sigma_N = \left(N + \frac{1}{2}\right)\pi, \quad \frac{\pi}{2} \leq \sigma_k \leq (2N + 1)\pi \quad \text{for } k = 1, 2, \ldots, N.
\]  

The \(L^2\)-norm relations between a piecewise linear function \(\hat{\psi} \in \mathcal{V}_N\) and its polynomial interpolation \(T_N \hat{\psi}\) can be shown as, using Proposition 4.7 and (7.6),

\[
c \| \hat{\psi} \|_0 \leq \| T_N \hat{\psi} \|_0 \leq C_N \| \hat{\psi} \|_0
\]  

where \(c\) is an absolutely positive constant and \(C_N\) is dependent only on \(N\) not on \(h\).
For functions \( v_j \in \mathcal{V}_{N_j} \) and \( u \in \mathcal{V}_{N}^h \), with the notation (4.8), from (7.7) applying to (4.9) we have
\[
c \| u \|_0 \leq \| T_N^h u \|_0 \leq C_N \| u \|_0,
\]
where \( c \) is an absolutely positive constant and \( C_N \) is dependent only on \( N_j \), which is the degree of local polynomial on \( j \)th subelement, but not on \( h \).

Finally, the two-dimensional preconditioning result can be verified using min-max theorem with (7.3)–(7.5) and (7.8). Hence we omit its proof.

**Theorem 7.1.** The eigenvalues \( \{ \lambda_{ji} \}_{i=1}^{d_N} \) of \( \tilde{C}^{-1} \tilde{S} \) are positive real and there are two positive constants \( c \) and \( C_N \) which are independent of mesh sizes \( h_j^x \) and \( h_j^y \) such that
\[
c \leq \lambda_{ji} \leq C_N.
\]

The above theorem implies that the condition numbers of \( \tilde{C}^{-1} \tilde{S} \) are independent of the mesh sizes \( h_j^x, h_j^y \).

**Remark 7.2.** For the case \( \beta = 0 \) in (2.1), high-order and finite element discretizations in 2D case give the tensor representations \( M^x \otimes S^x + S^x \otimes M^x \) and \( M^y \otimes S^y + S^y \otimes M^y \), respectively.

Note the discretization for this simple Laplace case becomes
\[
((M^x \otimes S^x) U, U) = ((M^y \otimes S^y) U, U) + ((S^x \otimes M^x) U, U).
\]

Therefore, in order to check the \( N \)-dependency of the preconditioner, it is enough to consider
\[
(M^y \otimes S^y)^{-1}(M^x \otimes S^x),
\]
which is \( (M^y)^{-1} M^x \otimes (S^y)^{-1} S^y \). This means that it reveals the \( N \)-dependency of the mass matrix even for the case \( \beta = 0 \).

### 7.1. Numerical computations for \( \tilde{C}^{-1} \tilde{S} \)

Consider two-dimensional problem (2.1)–(2.2) with \( \beta = 2 \) and the boundary (2.3). For the computations, the matrix \( \tilde{S} \) can be replaced by an equivalent \( \tilde{S} \) using LGL quadrature rule. That is, (7.1) can be approximated by
\[
B(u, u) \sim \langle u_x, u_x \rangle_{\eta^{2x}} + \langle u_y, u_y \rangle_{\eta^{2y}} + 2\langle u, u \rangle_{\eta^{2x}} \quad \text{for } u \in [P^h_{N}]^2.
\]

The matrix representation of the right side of (7.9) becomes
\[
\tilde{S} = M^x_T W_x M_x \otimes \tilde{G}_x + M^y_T W_y M_y,
\]
where
\[
\tilde{G}_i := D_i^T W_i D_i + M^x_i T W_i M_x, \quad i = x, y.
\]

For simplicity of computations, we consider the cases of same degrees \( N = N_j^x = N_j^y \) of polynomials in subrectangles whose mesh sizes \( h = h_j^x = h_j^y \) (equivalently \( M = M^x = M^y \)) are uniform; \( N_j^x = N_j^y = 4, 6, 8, 10 \) and \( M^x = M^y = 2, 4 \). The distributions of the eigenvalues of the preconditioned matrix \( \tilde{C}^{-1} \tilde{S} \) for the each case are provided in Fig. 3.

According to computations, the preconditioned matrices \( \tilde{C}^{-1} \tilde{S} \) have condition numbers, which are independent of the number of subelements (or the mesh sizes) and that grow slowly with the degrees of polynomials in Table 2 for \( N \leq 10 \) and \( M = 2, 4 \).

We have mentioned about the \( N \)-dependence of condition numbers in previous section which are caused by the effects of two one-dimensional mass matrices \( M^x \) and \( M^y \). These phenomena shown in Fig. 4 can be compared with Fig. 2. Actually, for a single element Fig. 4 reveals that the condition numbers are behaved likely as
\[
2.9 N^{1/3} \leq \kappa (\tilde{C}^{-1} \tilde{S}) \leq 3.4 N^{1/3} \quad \text{for } N \leq 40.
\]

That is, it seems to grow approximately with \( N^{1/3} \). The right of Fig. 4 presents the behaviors of the largest and smallest eigenvalues of \( \tilde{C}^{-1} \tilde{S} \). We can see that the graph of the smallest eigenvalues approaches a horizontal asymptote and
Table 2
Condition numbers of $\hat{C}^{-1}\tilde{S}$ (condition numbers of $\tilde{S}$)

<table>
<thead>
<tr>
<th>$N$</th>
<th>$M$</th>
<th>Condition number of $\hat{C}^{-1}\tilde{S}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>2</td>
<td>4.55394</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(142.48754)</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>4.51826</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(502.13872)</td>
</tr>
<tr>
<td>6</td>
<td>5.30036</td>
<td>(402.02016)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(1512.15378)</td>
</tr>
<tr>
<td>8</td>
<td>6.33057</td>
<td>(843.39633)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(3225.32215)</td>
</tr>
<tr>
<td>10</td>
<td>7.02570</td>
<td>(1526.01090)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(5860.43274)</td>
</tr>
</tbody>
</table>

Fig. 3. Spectrum of $\hat{C}^{-1}\tilde{S}$ in 2D with (a) $M^x = M^y = 2$, $N_j^x = N_j^y = 4$, (b) $M^x = M^y = 4$, $N_j^x = N_j^y = 6$, (c) $M^x = M^y = 2$, $N_j^x = N_j^y = 8$, (d) $M^x = M^y = 4$, $N_j^x = N_j^y = 10$.

Fig. 4. Behaviors of condition numbers/eigenvalues of $\hat{C}^{-1}\tilde{S}$ and $N^{1/3}$.

the approximate growth of the largest eigenvalues is on the order of $N^{1/3}$. The left of Fig. 5 shows the ratios of the condition numbers for $(M + 2, N)$ to the ones for $(M, N)$ for each fixed $N = 2, 4, 6$, which are converge to 1 as expected above. For a single element, the ratios of the condition numbers for $N + 4$ to the ones for $N$ are shown in the right of Fig. 5 with the 1D case. Table 3 provides the condition numbers of $\hat{C}^{-1}\tilde{S}$ for $N \leq 40$ with a single element $M = 1$. 
Fig. 5. Behaviors of ratios for condition numbers of $\hat{C}^{-1}\hat{S}$.

### Table 3
Condition numbers of $\hat{C}^{-1}\hat{S}$ for a single element

<table>
<thead>
<tr>
<th>$N$</th>
<th>4</th>
<th>16</th>
<th>28</th>
<th>40</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\kappa(\hat{C}^{-1}\hat{S})$</td>
<td>4.69662</td>
<td>8.28799</td>
<td>9.86249</td>
<td>11.18181</td>
</tr>
</tbody>
</table>

7.2. Numerical computations for $\hat{P}^{-1}\hat{K}$

In this subsection, we test an optimal preconditioner for two-dimensional case in [23] numerically. Now consider the matrix system (7.2) associated with the high-order element method represented in the form:

$$\hat{S}U = M^s f,$$

where $M^s$, $\hat{S}$ are the matrices defined in (5.4), (7.2), respectively, and $f$ is the coefficients vector with respect to the basis $\Phi_\mu$ of $[P^N_{h,N}]^2$. Taking into account the mass matrix $M^s$, we test the suggested preconditioner for $\hat{K} := (M^s)^{-1}\hat{S}$ or $\tilde{K} := (M^s)^{-1}\tilde{S}$, where $\tilde{S}$ is the matrix defined in (7.10),

$$\hat{P} := (M^f)^{-1}\hat{C}$$

where $M^f$, $\hat{C}$ are the matrices defined in (5.4), (7.4), respectively.

The preconditioning effects occurred by $\hat{P}$ are represented at Fig. 6 which shows that the condition numbers of $\hat{P}^{-1}\hat{K}$ are bounded independently of the polynomial degrees and mesh sizes numerically. This optimality phenomenon should be studied further analytically.

8. Conclusion

We analyze the nonweighted $H^1$ or $L^2$ equivalences of piecewise linear functions at CGL nodes and their interpolants at same nodes. To provide the numeric demonstrations, the nonweighted integral is approximated by the LGL quadrature using LGL nodes.

As we have shown that the condition numbers of the preconditioned systems are uniformly bounded in one-dimensional case, it is possible to find multigrid algorithms for solving elliptic boundary value problems with high-order discretizations based on CGL nodes. Since CGL nodes have the nested property, it makes adaptive $p$-refinement more straightforward. In two-dimensional case, it is proved that condition numbers are bounded independently of mesh sizes, but those depend on $N$. The numeric results show that they seem to grow approximately with $N^{1/3}$. Unfortunately, we do not provide these analytically. Further, an optimal preconditioner in [23] tested in Subsection 7.2 is also required to be analyzed. Note that such an optimal preconditioner for second-order boundary value problems including a linear elasticity equation were studied numerically using LGL nodes for spectral collocation methods in [23].
Fig. 6. Condition numbers of $\tilde{K}$ (left) and of $\tilde{P}^{-1}\tilde{K}$ (right).

Acknowledgements

The authors deeply thanks to Prof. Parter for his very careful reading and suggestions and also thanks to Dr. Fischer for valuable communications on this topic. We also deeply appreciate the referee’s suggestions for improving this paper.

Appendix A. Differentiation and mass matrices

The Chebyshev Lagrange interpolation polynomials using the CGL nodes give the identity matrix when it is evaluated at the CGL nodes, but it cannot be the identity matrix at the LGL nodes. It is because the representation nodes are different from the computational nodes. Hence the simple representation for such a matrix will be required. It is also known that the differentiation matrix for Chebyshev Lagrange interpolation polynomials at the CGL nodes is written explicitly (see [1,3] and [10] for example), but which cannot be used for the computation at the LGL quadrature. So, we need the matrices $M$ and $D$ of (5.8) explicitly. Now, we will find them for a single element and the arguments below can be extended to any odd degrees of polynomials via a minor modification. Hence we consider the case of even degree $N$ of a polynomial.

Theorem A.1. Let $M = \{M_{ij}\}$ and $D = \{D_{ij}\}$ be mass and differentiation matrices, respectively, defined in (5.8). Then the matrices can be written as

$$M_{ij} = \begin{cases} 1, & i = j = 0, i = j = N, \\ 1, & i = j = N/2, \\ 0, & i = 0, j = 1, 2, \ldots, N, \\ 0, & i = N, j = 0, 1, \ldots, N - 1, \\ (-1)^{i+1}(1 - (\eta_i^2)^2)U_{N-1}(\eta_i)^2, & \frac{d_j}{\eta_i^2 - \eta_j^2}, \\ & \text{otherwise,} \end{cases} \quad (A.1)$$

and
The other cases can be obtained from (A.4). These arguments give the explicit representation for $M$

$$D_{ij} = \begin{cases} \frac{2N^2 + 1}{6}, & i = j = 0, \\ \frac{2N^2 + 1}{6}, & i = j = N, \\ 0, & i = j = N/2, \\ \frac{(-1)^{j+1}}{d_j} \frac{2}{1 + \eta_j^c}, & i = 0, \ j = 1, 2, \ldots, \ N, \\ \frac{(-1)^{j+1}}{d_j} \frac{-2}{1 - \eta_j^c}, & i = N, \ j = 0, 1, \ldots, \ N - 1, \\ \frac{(-1)^{j+1}}{d_j N} \frac{1}{\eta_i^c - \eta_j^c} \left[ T_{N+1}(\eta_i^c) - \eta_i^c T_N(\eta_i^c) \right] - NT_N(\eta_i^c) - U_N(\eta_i^c) \right], & \text{otherwise}, \end{cases}$$

where $d_0 = d_N = 2$, $d_j = 1$ ($1 \leq j \leq N - 1$), $\eta_i^c$, $\eta_j^c$ are CGL, LGL points respectively, and

$$T_N(t) = \cos\left( N \cos^{-1} t \right)$$

is Chebyshev’s polynomial of the first kind and

$$U_N(t) = \frac{\sin((N + 1) \cos^{-1} t)}{\sin(\cos^{-1} t)}$$

is Chebyshev’s polynomial of the second kind.

**Proof.** Note that Chebyshev Lagrange polynomial is

$$\phi_j(t) = \frac{(-1)^{j+1}}{d_j N^2} \frac{(1 - t^2)T_N(t)}{t - \eta_j^c} \quad \text{for } j = 0, 1, \ldots, N, \quad (A.2)$$

where $d_0 = d_N = 2$, $d_j = 1$ for $j = 1, 2, \ldots, N - 1$ and recall the relations (see [11] for example)

$$T_k(t) = U_k(t) - t U_{k-1}(t), \quad (A.3)$$

$$T_k'(t) = k U_{k-1}(t),$$

$$1 - t^2)T_k'(t) = k(T_k(t) - T_{k+1}(t)). \quad (A.4)$$

Also, note that $T_k'(\pm 1) = (\pm 1)^k k^2$ for $k = 0, \ldots, N$. Now, consider $M_{ij} = \phi_j(\eta_i^c)$ for $i, j = 0, 1, \ldots, N$. Since

$$\eta_i^c = \eta_j^c \quad \text{for } \begin{cases} i = j = 0, \\ i = j = N, \\ i = j = N/2, \end{cases}$$

so that $\phi_j(\eta_i^c) = \phi_j(\eta_j^c) = 1$ for $i = j = 0, \ N, \ N/2$. And $\eta_0^c = -1$ and $\eta_N^c = 1$ yield

$$M_{ij} = 0 \quad \text{for } \begin{cases} i = 0, \ j \neq 0, \\ i = N, \ j \neq N. \end{cases} \quad (A.6)$$

The other cases can be obtained from (A.4). These arguments give the explicit representation for $M$ in (A.1). Differentiating (A.2) and applying (A.3)–(A.5) in reverse order, we get

$$\phi_j'(t) = \frac{(-1)^{j+1}}{d_j N} \frac{1}{t - \eta_j^c} \left[ \frac{T_{N+1}(t) - \eta_j^c T_N(t)}{t - \eta_j^c} - NT_N(t) - U_N(t) \right).$$

Using (A.6), in the case of $i = j = 0, \ N, \ N/2$, it is given by Chebyshev differential matrix (see [1,3]). For the cases $\eta_i^c = \pm 1$, that is, $i = 0$ or $i = N$, using $U_N(\pm 1) = N + 1$, we have

$$D_{ij} = \begin{cases} \frac{(-1)^{j+1}}{d_j} \frac{2}{1 + \eta_j^c} & \text{for } i = 0, \ j = 1, \ldots, N, \\ \frac{(-1)^{j+1}}{d_j} \frac{-2}{1 - \eta_j^c} & \text{for } i = N, \ j = 0, \ldots, N - 1. \end{cases}$$
The other cases are just same as (A.7). It completes the proof. □

References